King Fahd University of Petroleum and Minerals
College of Computer Sciences and Engineering
Information and Computer Science Department

ICS 253: Discrete Structures I
Summer semester 2017-2018
Major Exam #1, Saturday July 14, 2018
Time: **100** Minutes

Name: ____

ID#: ________________________________

Instructions:
1. The exam consists of 8 pages, including this page, containing 7 questions.
2. Answer all 7 questions. *Show all the steps*.
3. Make sure your answers are **clear** and **readable**.
4. The exam is closed book and closed notes. No calculators or any helping aides are allowed. Make sure you turn off your mobile phone and keep it in your pocket.
5. If there is no space on the front of the page, use the back of the page.

Question	Maximum Points	Earned Points
1	15	
2	10	
3	15	
4	25	
5	15	
6	10	
7	10	
Total	**100**	

Rules of Inference:

$p \rightarrow (p \vee q)$	Addition	$[\neg q \wedge (p \rightarrow q)] \rightarrow \neg p$	Modus Tollens
$(p \wedge q) \rightarrow p$	Simplification	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism
$[(p) \wedge (q)] \rightarrow (p \wedge q)$	Conjunction	$[(p \vee q) \wedge \neg p] \rightarrow q$	Disjunctive syllogism
$[p \wedge (p \rightarrow q)] \rightarrow q$	Modus Ponens	$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$	Resolution
$\forall x P(x) \rightarrow P(a)$ for all a	Universal Instantiation	$\exists x P(x) \rightarrow P(a)$ for some a	Existential Instantiation
$P(a)$ for all a $\rightarrow \forall x P(x)$	Universal Generalization	$P(a)$ for some a $\rightarrow \exists x P(x)$	Existential Generalization

Q1: [15 points] Answer the following questions.

a) [8 points] Let p, q, and r be the propositions
p: Belgian players have a lot of practice.
q: The coach has an effective strategy.
r: The Belgian team wins the match.

Write these propositions using p and q and logical connectives (including negations).

i. Belgian players have a lot of practice, but the coach does not have an effective strategy.

$$p \wedge \neg q$$

ii. It is necessary and sufficient that the Belgian players have a lot of practice, for the Belgian team to win the match.

$$p \Leftrightarrow r$$

iii. Unless the coach has an effective strategy, the Belgian team would not win the match.

$$\neg q \rightarrow \neg r$$

b) [7 points] Without using a truth table, prove that $[(p \vee q) \wedge \neg p] \rightarrow q$ is a tautology.

$$[(p \vee q) \wedge \neg p] \rightarrow q \Leftrightarrow \neg ((p \vee q) \wedge \neg p) \vee q$$
$$\Leftrightarrow ((\neg p \wedge \neg q) \vee p) \vee q$$
$$\Leftrightarrow ((\neg p \vee p) \wedge (\neg q \vee p)) \vee q$$
$$\Leftrightarrow (T \wedge (\neg q \vee p)) \vee q$$
$$\Leftrightarrow (\neg q \vee p) \vee q$$
$$\Leftrightarrow (\neg q \vee q) \vee p$$
$$\Leftrightarrow T \vee p \Leftrightarrow T.$$

Q2: [10 points] Recall the island of the knights and knaves, where knights always tell the truth and knaves always lie. You meet three inhabitants: Alice, Rex and Bob. Determine, if possible, who is knight and who is knave, if they said the following: Alice says that Rex is a knave. Rex says that it's false that Bob is a knave. Bob claims, "I am a knight or Alice is a knight." Make sure you clearly justify your answer.

Alice : Rex is a knave
Rex : $\neg$(Bob is a knave)
Bob : Bob is knight or Alice is knight

Assume Alice is knight
Then Rex is a knave and Bob is a knave
But then from Bob, Bob is knave and Alice is knave, which is a contradictn.

Assume Alice is knave.
Then, Rex is a knight,
Hence, Bob is a knight,
$\therefore$ Bob's statement is true since Bob is knight.

$\therefore$ Alice is knave & Bob & Rex are knights.

Q3: [15 points] Steve would like to determine the relative salaries of three coworkers using two facts. First, he knows that if Fred is not the highest paid of the three, then Janice is. Second, he knows that if Janice is not the lowest paid, then Maggie is paid the most. Is it possible to determine the relative salaries of Fred, Maggie, and Janice from what Steve knows? If so, who is paid the most and who the least? Explain your reasoning.

Hint: Use truth tables and discard illogical combinations!

Consider $p \equiv$ Fred is the highest paid.
$q \equiv$ Janice is the highest paid.
$r \equiv$ Janice is the lowest paid.
$s \equiv$ Maggie is the highest paid.

One possible solution is to the form the following truth table:

#	p	q	r	s	$\neg p$	$\neg r$	$\neg p \to q$	$\neg r \to s$	$(\neg p \to q) \wedge (\neg r \to s)$
1	T	T	T	T	F	F	T	T	T
2	T	T	T	F	F	F	T	T	T
3	T	T	F	T	F	T	T	T	T
4	T	T	F	F	F	T	T	F	F
5	T	F	T	T	F	F	T	T	T
6	T	F	T	F	F	F	T	T	T
7	T	F	F	T	F	T	T	T	T
8	T	F	F	F	F	T	T	F	F
9	F	T	T	T	T	F	T	T	T
10	F	T	T	F	T	F	T	T	T
11	F	T	F	T	T	T	T	T	T
12	F	T	F	F	T	T	T	F	F
13	F	F	T	T	T	F	F	T	F
14	F	F	T	F	T	F	F	T	F
15	F	F	F	T	T	T	F	T	F
16	F	F	F	F	T	T	F	F	F

Now, we check the possibility of the scenario in each highlighted row:
Rows 1 ... 3 cannot happen because Fred and Janice cannot be both paid the highest.
Rows 5 and 7 cannot happen because Fred and Maggie cannot be both paid the highest.
Rows 9 and 11 cannot happen because Janice and Maggie cannot be both paid the highest.
Now, we are left with Row 6 and Row 10.
Row 6 says that Fred is the Highest, Janice is the lowest and neither Maggie nor Janice is the highest. Hence, the ordering from highest paid to lowest paid is Fred, Maggie, and Janice, respectively.
Row 10 cannot happen because Janice cannot be the highest paid and the lowest paid at the same time.
Therefore, and based on Steve's facts, Row 6 gives us the right ordering:

$$Fred > Maggie > Janice$$

Another solution:

Let p, q, r, s be as before. Then, if p is true, i.e. Fred is the highest, then q & s have to be false. Also if Janice is the highest (i.e. q is true) then p, r and s have to be all false.

∴ The truth table reduces to only rows with numbers 6, 8, 12, 13 and 15 only.

Therefore, it is sufficient that your truth table only contain the above 5 rows, of which one and only one row has the two facts being correct.

Another solution:

Instead of using simple propositions that can be true or false use the following predicates:

$F(x)$: Fred is paid x salary.

$M(x)$: Maggie is paid x salary.

$J(x)$: Janice is paid x salary.

where the domain of x is {highest, middle, lowest}

= {H, M, L}.

Then, our truth table has the following possible cases only:

$F(x)$	$J(x)$	$M(x)$	① $\neg F(H) \to J(H)$	② $\neg J(L) \to M(H)$	① $\wedge$ ②
L	M	H	F	T	F
L	H	M	T	F	F
M	L	H	F	T	F
M	H	L	T	F	F
H	L	M	T	T	T
H	M	L	T	F	F

Q4: [25 points] Let $F(x, y)$ be the statement "x can fool y," and $K(z)$ be the statement "z is a KFUPM student." where the domain of the variables x, y, and z consists of all people in the world. Use quantifiers and predicates to express each of these statements, where no negation is outside a quantifier or an expression involving logical connectives. You are **not** allowed to use "∃!".

a) [5 points] Ahmad, who is a student at KFUPM, cannot fool Salim.

$$K(\text{Ahmad}) \wedge \neg F(\text{Ahmad}, \text{Salim})$$

b) [5 points] KFUPM students can fool Lionel.

$$\forall x\,(K(x) \rightarrow F(x, \text{Lionel}))$$

c) [5 points] No one can fool any KFUPM student.

$$\neg(\exists x\,\exists y\,(K(y) \wedge F(x,y)))$$

$$\Leftrightarrow \quad \forall x\,\forall y\,(K(y) \rightarrow \neg F(x,y))$$

d) [5 points] There is a KFUPM student who was fooled by every other student at KFUPM.

$$\exists y\,(K(y) \wedge \forall x\,([K(x) \wedge (x \neq y)] \rightarrow F(x,y)))$$

e) [5 points] There is exactly one non-KFUPM student who can fool every KFUPM student.

$$\exists x\,(\neg K(x) \wedge [\forall y\,(K(y) \rightarrow F(x,y))] \wedge \neg\exists z\,(\neg K(z) \wedge (z \neq x) \wedge \forall w\,(K(w) \rightarrow F(z,w)))$$

$$\Leftrightarrow \exists x\,(\neg K(x) \wedge (\forall y\,(K(y) \rightarrow F(x,y))) \wedge \forall z\,([\forall w\,(K(w) \rightarrow F(z,w))] \rightarrow (K(z) \vee (z = x))))$$

Q5: [15 points] Answer the following questions.

a) (6 points) Determine the truth value of each of these statements if the domain of each variable consists of all real numbers. If it is false, prove it.

i. $\forall x\big(x \neq 0 \rightarrow \exists y(xy = 1)\big)$

True.

ii. $\forall x \exists y(x = y^2)$

False. $x = -1$. No y s.t. $y^2 = -1$ in $\mathbb{R}$

iii. $\exists y \forall x(x + y = 1)$

False. If such $y_0 \in \mathbb{R}$ exists, then $x = 1 - y_0$ is true for only a single value of x.

b) (5 points) Show that the hypotheses "It is not raining or Yvette has her umbrella," "Yvette does not have her umbrella or she does not get wet," and "It is raining or Yvette does not get wet" imply that "Yvette does not get wet."

Let $r \equiv$ It is raining.
$u \equiv$ Yvette has her umbrella.
$w \equiv$ Yvette gets wet.

1. $\neg r \vee u$
2. $\neg u \vee \neg w$
3. $r \vee \neg w$

4. $\therefore$ $\neg w$

1 & 2 & resolution $\rightarrow$ $\neg r \vee \neg w$ — 5

5 & 3 & resolution $\rightarrow$ $\neg w \vee \neg w$ — 6

6 & Idempotent rule $\rightarrow$ $\neg w$ which is the conclusion.

c) (4 points) Identify the error or errors in this argument that supposedly shows that if $\forall x(P(x) \vee Q(x))$ is true then $\forall x P(x) \vee \forall x Q(x)$ is true.

1.	$\forall x(P(x) \vee Q(x))$	Premise
2.	$P(c) \vee Q(c)$	Universal Instantiation from (1)
3.	$P(c)$	Simplification from (2)
4.	$\forall x P(x)$	Universal generalization from (3)
5.	$Q(c)$	Simplification from (2)
6.	$\forall x Q(x)$	Universal generalization from (5)
7.	$\forall x P(x) \vee \forall x Q(x)$	Conjunction from (4) and (6).

Step 3: Cannot do simplificatn from disjunction.

Step 5: Same reasoning of Step3.

Step 7: Conjunction from 4 & 6 leads to

$$\forall x P(x) \wedge \forall x (Q(x)).$$

Q6: [10 points] Prove that if *m* and *n* are integers and *mn* is even, then *m* is even or *n* is even.

Proof by contraposition.

Let $m, n \in \mathbb{Z}$, and assume that m is odd & n is odd.

i.e. $m = 2r+1$ & $n = 2s+1$ where $r, s \in \mathbb{Z}$.

Then, $mn = (2r+1)(2s+1)$

$= 4rs + 2r + 2s + 1$

$= 2(2rs + r + s) + 1$

Since $2rs + r + s \in \mathbb{Z}$

$2(2rs+r+s)+1$ is odd

i.e. mn is odd.

Q7: [10 points] Answer the following questions.

a) (2 points) List the elements of $\mathcal{P}(\{\Phi, \{a\}\})$ where $\mathcal{P}$ denotes the power set and Φ denotes the empty set.

$\{\Phi, \{\Phi\}, \{\{a\}\}, \{\Phi, \{a\}\}\}$

b) (4 points) For $A_i = [-i, i]$, where i is a positive integer and $[a, b]$ represents all real numbers x such that $a \leq x \leq b$.

i. Find $\bigcup_{i=1}^{\infty} A_i$

$= [-1, 1] \cup [-2, 2] \cup [-3, 3] \cup \ldots$

$= \mathbb{R}$

ii. Find $\bigcap_{i=1}^{\infty} A_i$

$= [-1, 1] \cap [-2, 2] \cap [-3, 3] \cap \ldots$

$= [-1, 1] = A_1.$

c) (4 points) Draw the Venn diagram for $(A \cap \bar{B}) - C$ of the sets A, B and C.